# On the algebras obtained by tensor product.

Elisabeth REMM *- Michel GOZE †

Université de Haute Alsace, F.S.T.
4, rue des Frères Lumière - 68093 MULHOUSE - France

**2000 Mathematics Subject Classification.** Primary 18D50, 17A30, 17Bxx, Secondary 17D25.

**Keywords.** Tensor product of algebras on quadratic algebras.

**Abstract**

Let $\mathcal{P}$ be a quadratic operad. We define an associated operad $\tilde{\mathcal{P}}$ such that for any $\mathcal{P}$-algebra $\mathcal{A}$ and $\tilde{\mathcal{P}}$-algebra $\mathcal{B}$, the algebra $\mathcal{A}\otimes\mathcal{B}$ is always a $\mathcal{P}$-algebra for the classical tensor product.

## 1 Introduction

Let $\mathcal{P}$ be a quadratic operad with one generating operation (i.e. the algebras on this operad have only one operation) and $\mathcal{P}^!$ its dual operad. It satisfies $\mathcal{P}^! = hom(\mathcal{P}, \mathcal{L}ie)$ where $\mathcal{L}ie$ is the quadratic operad corresponding to Lie algebras. For any $\mathcal{P}$-algebra $\mathcal{A}$ and $\mathcal{P}^!$-algebra $\mathcal{B}$, the vector space $\mathcal{A}\otimes\mathcal{B}$ is naturally provided with a Lie algebra product

$$\mu(a_1\otimes b_1, a_2\otimes b_2) = \mu_{\mathcal{A}}(a_1,a_2)\otimes\mu_{\mathcal{B}}(b_1,b_2) - \mu_{\mathcal{A}}(a_2,a_1)\otimes\mu_{\mathcal{B}}(b_2,b_1). \quad (1)$$

where $\mu_{\mathcal{A}}$ (resp. $\mu_{\mathcal{B}}$) is the multiplication of $\mathcal{A}$ (resp. $\mathcal{B}$). We deduce that the "natural" tensor product $\mu_{\mathcal{A}\otimes\mathcal{B}} = \mu_{\mathcal{A}}\otimes\mu_{\mathcal{B}}$ provides $\mathcal{A}\otimes\mathcal{B}$ with a Lie-admissible algebra structure.

In [2] we have defined special classes of Lie-admissible algebras with relations of definition determined by an action of the subgroups $G_i$ of the 3-degree symmetric group $\Sigma_3$. We obtain quadratic operads, denoted by $G_i-\mathcal{A}ss$ and in this family we find operads of Lie-admissible, associative, Vinberg and pre-Lie algebras. For these operads we have proved that for every $\mathcal{P}$-algebra $\mathcal{A}$ and $\mathcal{P}^!$-algebra $\mathcal{B}$ the tensor product $\mathcal{A}\otimes\mathcal{B}$ is a $\mathcal{P}$-algebra. This is not true for general nonassociative algebras and, in this sense, the $G_i$-associative algebras are the most regular kind of nonassociative algebras. For example if $\mathcal{P}$ is the operad of Leibniz algebras or of the nonassociative algebras associated to Poisson algebras [?], then the tensor product of a $\mathcal{P}$-algebra and $\mathcal{P}^!$-algebra is not a $\mathcal{P}$-algebra.

So we introduce a quadratic operad, denoted by $\tilde{\mathcal{P}}$, such that the tensor product of a $\mathcal{P}$-algebra with a $\tilde{\mathcal{P}}$-algebra is a $\mathcal{P}$-algebra. In case of $\mathcal{P} = \mathcal{L}ie$ or $G_i$-$\mathcal{A}ss$ then $\tilde{\mathcal{P}} = \mathcal{P}^!$ this explain the above remarks.

*corresponding author: e-mail: E.Remm@uha.fr
†M.Goze@uha.fr.

## 2 Nonassociative algebras and operads

We assume in this work that $\mathcal{P}$ is a quadratic operad with one generating operation. Then it is defined from the free operad $\Gamma(E)$ generated by a $\Sigma_2$-module $E$ placed in arity 2, and an ideal $(R)$ generated by a $\Sigma_3$-invariant subspace $R$ of $\Gamma(E)(3)$. Our hypothesis implies that the $\Sigma_2$-module $E$ is generated by one element (i.e. algebras over this operad are algebras with one operation). Recall that, if we consider an operation with no symmetry, the $\mathbb{K}$-vector space $\Gamma(E)(2)$ is 2-dimensional with basis $\{x_1 \cdot x_2, x_2 \cdot x_1\}$ and $\Gamma(E)(3)$ is the 12-dimensional $\mathbb{K}$-vector space generated by $\{x_i \cdot (x_j \cdot x_k), (x_i \cdot x_j) \cdot x_k\}$ with $i \neq j \neq k \neq i$, $i, j, k \in \{1, 2, 3\}$. We have a natural action of $\Sigma_3$ on $\Gamma(E)(3)$ given by:

$$\begin{array}{cccl} \Sigma_3 & \times & \Gamma(E)(3) & \rightarrow \Gamma(E)(3) \\ (\sigma & , & X) & \mapsto \sigma(X) \end{array}$$

where

$$\begin{array}{l} \sigma(x_i \cdot (x_j \cdot x_k)) = x_{\sigma^{-1}(i)} \cdot (x_{\sigma^{-1}(j)} \cdot x_{\sigma^{-1}(k)}) \\ \sigma((x_i \cdot x_j) \cdot x_k) = (x_{\sigma^{-1}(i)} \cdot x_{\sigma^{-1}(j)}) \cdot x_{\sigma^{-1}(k)}). \end{array}$$

We denote by $\mathcal{O}(X)$ the orbit of $X$ associated with this action and by $\mathbb{K}(\mathcal{O}(X))$ the $\Sigma_3$-invariant subspace of $\Gamma(E)(3)$ generated by $\mathcal{O}(X)$. More generally, if $X_1, \cdots, X_k$ are vectors in $\Gamma(E)(3)$, we denote by $\mathbb{K}(\mathcal{O}(X_1, .., X_k))$ the $\Sigma_3$-invariant subspace of $\Gamma(E)(3)$ generated by $\mathcal{O}(X_1) \cup \cdots \cup \mathcal{O}(X_k)$.

**Definition 1** *We say that a $\Sigma_3$-invariant subspace $F$ of $\Gamma(E)(3)$ is of rank $k$ if there exists $X_1, ..., X_k \in \Gamma(E)(3)$ linearly independent such that $F = \mathbb{K}(\mathcal{O}(X_1, ..., X_k))$ and for every $p < k$ and $Y_1, ...Y_p \in F$ we have $\mathbb{K}(\mathcal{O}(Y_1, ..., Y_p)) \neq F$.*

If $\mathcal{P}$ is a quadratic operad with one generating operation, its module of relations $R$ is an invariant subspace of $\Gamma(E)(3)$. We will say that $\mathcal{P}$ is of rank $k$ if and only if $R$ is of rank $k$.

From the action of $\Sigma_3$ on $\Gamma(E)(3)$ we define linear maps on this module as follows. Let $\mathbb{K}[\Sigma_3]$ be the group algebra of $\Sigma_3$ that is the vector space of all finite linear combinations of elements of $\Sigma_3$ with coefficients in $\mathbb{K}$ hence of all elements of the form $v = a_1\sigma_1 + a_2\sigma_2 + ... + a_6\sigma_6$. If $v = \Sigma a_i \sigma_i \in \mathbb{K}[\Sigma_3]$, let $\Psi_v$ be given by

$$\Psi_v(X) = \Sigma a_i \sigma_i(X).$$

Then an invariant subspace $F$ of $\Gamma(E)(3)$ is stable for every $\Psi_v$.

Let $(\mathcal{A}, \mu)$ be a $\mathcal{P}$-algebra. This means that $(\mathcal{A}, \mu)$ is a nonassociative algebra (by nonassociative algebra we mean algebras with non necessarily associative multiplication). We consider the maps $A^L(\mu) = \mu \circ (\mu \otimes Id)$ and $A^R(\mu) = \mu \circ (Id \otimes \mu)$. Then the associator of $\mu$ is written $A(\mu) = A^L(\mu) - A^R(\mu)$. For each vector $v \in \mathbb{K}[\Sigma_3]$ we define a linear map on $\mathcal{A}^{\otimes 3}$ denoted by $\Phi_v^{\mathcal{A}}$ and given by

$$\begin{array}{ccl} \Phi_v^{\mathcal{A}} : \mathcal{A}^{\otimes 3} & \rightarrow & \mathcal{A}^{\otimes 3} \\ (x_1 \otimes x_2 \otimes x_3) & \mapsto & \sum a_i(x_{\sigma^{-1}(1)} \otimes x_{\sigma^{-1}(2)} \otimes x_{\sigma^{-1}(3)}) \end{array}$$

The multiplication $\mu$ satisfies relations of the following type

$$A^L(\mu) \circ \Phi_v^{\mathcal{A}} - A^R(\mu) \circ \Phi_{v'}^{\mathcal{A}} = 0, \quad (2)$$

where $v = \Sigma a_i \sigma_i, v' = \Sigma a'_i \sigma_i \in \mathbb{K}[\Sigma_3]$.

Such a relation defines the module $R$ of relations of $\mathcal{P}$. In fact $R$ is the $\Sigma_3$-submodule of $\Gamma(E)(3)$ is generated by the vectors

$$\sigma_j(\Psi_v(x_1 \cdot (x_2 \cdot x_3)) - \Psi_{v'}((x_1 \cdot x_2) \cdot x_3))$$

for every $\sigma_j \in \Sigma_3$.

Let us note that if $\mathcal{P}$ is of rank 1, a $\mathcal{P}$-algebra is given by one multiplication satisfying only one relation of type (2).

**Proposition 2** *Let $\mathcal{P}$ be a quadratic operad with one generating operation such that the $\Sigma_3$-submodule $R$ of relations is generated by vectors of the following type :*

$$\Sigma_{i=1}^6 a_i^l \sigma_i((x_1 \cdot x_2) \cdot x_3 - x_1 \cdot (x_2 \cdot x_3))$$

*for $l = 1, ..., k$ and $\sigma_i \in \Sigma_3$. Then $\mathcal{P}$ is of rank 1.*

*Proof.* In fact, the $\Sigma_3$-invariant subspace of $\Gamma(E)(3)$ generated by

$$((x_1 \cdot x_2) \cdot x_3 - x_1 \cdot (x_2 \cdot x_3))$$

is isomorphic to $\mathbb{K}[\Sigma_3]$. This isomorphism is given by:

$$\Sigma a_i \sigma_i((x_1 \cdot x_2) \cdot x_3 - x_1 \cdot (x_2 \cdot x_3)) \longrightarrow \Sigma a_i \sigma_i.$$

We have seen in [3] that for every $\Sigma_3$-invariant subspace $F$ of $\mathbb{K}[\Sigma_3]$, there is a vector $v \in \mathbb{K}[\Sigma_3]$ such that $F = F_v = \mathbb{K}(\mathcal{O}(v))$ where $\mathcal{O}(v)$ is the orbit of $v$ corresponding to the natural action of $\Sigma_3$ on $\mathbb{K}[\Sigma_3]$. We deduce that the rank is 1.

In the following examples we will recall the definition of the operads $G_i$-$\mathcal{A}ss$ and define some quadratic operads of rank one associated to some classes of nonassociative algebras.

**Examples.**

**1. The operads $G_i - \mathcal{A}ss$**

For $i, j, k \in \{1, 2, 3\}$ and $i \neq j \neq k \neq i$ we denote by $\tau_{ij}$ the transposition $(i, j)$ and $c_1, c_2$ the cycles $(1, 2, 3)$ and $(1, 3, 2)$. The subgroups of $\Sigma_3$ are $G_1 = \{Id\}, G_2 =< \tau_{12} >, G_3 =< \tau_{23} >, G_4 =< \tau_{13} >$, $G_5 =< c_1 >$, $G_6 = \Sigma_3$ where $< g >$ denotes the subgroup generated by $g$. Each one of these subgroups $G_i$ defines an invariant submodule $R_i$ of $\Gamma(E)(3)$ of rank 1. In fact consider the vector $X = x_1 \cdot (x_2 \cdot x_3) - (x_1 \cdot x_2) \cdot x_3$ of $\Gamma(E)(3)$ and if $G_i$ is a subgroup of $\Sigma_3$, we define the vector $X_i$ of $\Gamma(E)(3)$ by

$$X_i = \sum_{\sigma \in G_i} (-1)^{\epsilon(\sigma)} \sigma(X)$$

where $\epsilon(\sigma)$ is the sign of the permutation $\sigma$. Let $R_i$ be the subspace $R_i = \mathbb{K}(O(V_i))$. Then

$$
\begin{aligned}
R_1 = \ & Vect_{\mathbb{K}}\left\{(x_i \cdot x_j)\cdot x_k - x_i\cdot(x_j\cdot x_k)\right\},\\
R_2 = \ & Vect_{\mathbb{K}}\left\{(x_i \cdot x_j)\cdot x_k - x_i\cdot(x_j\cdot x_k) - (x_j\cdot x_i)\cdot x_k + x_j\cdot(x_i\cdot x_k)\right\},\\
R_3 = \ & Vect_{\mathbb{K}}\left\{(x_i \cdot x_j)\cdot x_k - x_i\cdot(x_j\cdot x_k) - (x_i\cdot x_k)\cdot x_j + x_i\cdot(x_k\cdot x_j)\right\},\\
R_4 = \ & Vect_{\mathbb{K}}\left\{(x_i \cdot x_j)\cdot x_k - x_i\cdot(x_j\cdot x_k) - (x_k\cdot x_j)\cdot x_i + x_k\cdot(x_j\cdot x_i)\right\}\\
R_5 = \ & Vect_{\mathbb{K}}\{(x_i \cdot x_j)\cdot x_k - x_i\cdot(x_j\cdot x_k) + (x_j\cdot x_k)\cdot x_i - x_j\cdot(x_k\cdot x_i)\\
& +(x_k\cdot x_i)\cdot x_j - x_k\cdot(x_i\cdot x_j)\\
R_6 = \ & Vect_{\mathbb{K}}\{(x_1 \cdot x_2)\cdot x_3 - x_1\cdot(x_2\cdot x_3) + (x_2\cdot x_3)\cdot x_1 - x_2\cdot(x_3\cdot x_1)\\
& +(x_3\cdot x_1)\cdot x_2 - x_3\cdot(x_1\cdot x_2) - (x_2\cdot x_1)\cdot x_3 + x_2\cdot(x_1\cdot x_3) - (x_3\cdot x_2)\cdot x_1\\
& +x_3\cdot(x_2\cdot x_1) - (x_1\cdot x_3)\cdot x_2 + x_1\cdot(x_3\cdot x_2)\}
\end{aligned}
$$

**Definition 3** *The quadratic operad $G_i$-$\mathcal{A}ss$ is the quadratic operad $\Gamma(E)/(R)$ where $(R)(3) = R_i$.*

Some of these operads are wellknown:
- $G_1$-$\mathcal{A}ss = \mathcal{A}ss$,
- $G_2$-$\mathcal{A}ss = \mathcal{V}inb$,
- $G_3$-$\mathcal{A}ss = \mathcal{P}re - Lie$.

Let us note that the $G_6$-$\mathcal{A}ss$-algebras are the Lie-admissible algebras that is if $\mu$ is the product of such an algebra then

$$[x,y] = \mu(x,y) - \mu(y,x)$$

is a product of Lie algebra. As $G_i$ is a subgroup of $G_6 = \Sigma_3$ any $G_i$-$\mathcal{A}ss$-algebra is Lie-admissible. For a general study of the operad $G_6$-$\mathcal{A}ss = \mathcal{L}ie\mathcal{A}dm$ and $G_2$-$\mathcal{A}ss = \mathcal{V}inb$ see [2].

Let $(G_i$-$\mathcal{A}ss)^!$ be the quadratic dual operad of $G_i$-$\mathcal{A}ss$. We denote by $(R_i)^!$ the submodule of $\Gamma(E)(3)$ defining $(G_i$-$\mathcal{A}ss)^!$. It is the orthogonal of $R_i$ with respect to the inner product on $\Gamma(E)(3)$ given by

$$\langle i\cdot(j\cdot k), i\cdot(j\cdot k)\rangle = \epsilon\begin{pmatrix} 1 & 2 & 3\\ i & j & k\end{pmatrix}, \quad \langle (i\cdot j)\cdot k, (i\cdot j)\cdot k\rangle = -\epsilon\begin{pmatrix} 1 & 2 & 3\\ i & j & k\end{pmatrix}$$

where $\epsilon(\sigma)$ is the sign of the permutation $\sigma$. Then we have

$$
\begin{aligned}
(R_1)^! = \ & R_1\\
(R_2)^! = \ & Vect_{\mathbb{K}}\left\{(x_i \cdot x_j)\cdot x_k - x_i\cdot(x_j\cdot x_k); (x_i\cdot x_j)\cdot x_k - (x_j\cdot x_i)\cdot x_k\right\}\\
(R_3)^! = \ & Vect_{\mathbb{K}}\left\{(x_i \cdot x_j)\cdot x_k - x_i\cdot(x_j\cdot x_k); (x_i\cdot x_j)\cdot x_k) - (x_i\cdot x_k)\cdot x_j\right\}\\
(R_4)^! = \ & Vect_{\mathbb{K}}\left\{(x_i \cdot x_j)\cdot x_k - x_i\cdot(x_j\cdot x_k); (x_i\cdot x_j)\cdot x_k - (x_k\cdot x_j)\cdot x_i\right\}\\
(R_5)^! = \ & Vect_{\mathbb{K}}\left\{(x_i \cdot x_j)\cdot x_k - x_i\cdot(x_j\cdot x_k); (x_i\cdot x_j)\cdot x_k - (x_j\cdot x_k)\cdot x_i\right\}\\
(R_6)^! = \ & Vect_{\mathbb{K}}\{(x_i \cdot x_j)\cdot x_k - x_i\cdot(x_j\cdot x_k); (x_i\cdot x_j)\cdot x_k - (x_j\cdot x_i)\cdot x_k\\
& (x_i\cdot x_j)\cdot x_k - (x_i\cdot x_k)\cdot x_j\}
\end{aligned}
$$

**Proposition 4** *For $i = 1$, the operad $(G_1$-$\mathcal{A}ss)^! = \mathcal{A}ss$ is if rank $1$. For $2 \leq i \leq 6$, the operads $(G_i$-$\mathcal{A}ss)^!$ are of rank $2$.*

*Proof.* The case $i = 1$ is trivial (it is also a consequence of Proposition 2).
For $i = 2, 3, 4$ and 5, the rank of $(R_i)^!$ is 2. We denote by $v_i^j, j = 1, 2$ the generators of $(R_i)^!$. Then if $\mathcal{B}$ is a $(G_i\text{-}\mathcal{A}ss)^!$-algebra the multiplication $\mu_{\mathcal{B}}$ satisfies

$$\begin{array}{l} 1)\ \mu_{\mathcal{B}}(\mu_{\mathcal{B}} \otimes Id) - \mu_{\mathcal{B}}(Id \otimes \mu_{\mathcal{B}}) = 0 \\ 2)\ \mu_{\mathcal{B}}(\mu_{\mathcal{B}} \otimes Id) - \mu_{\mathcal{B}}(Id \otimes \mu_{\mathcal{B}} \circ \Phi_{\sigma_i}) = 0 \end{array}$$

with

$$\begin{cases} \sigma_2 = \tau_{12} \\ \sigma_3 = \tau_{23} \\ \sigma_4 = \tau_{13} \\ \sigma_5 = c_1 \quad \text{or} \quad c_2 \end{cases}$$

For $i = 6$, the space $(R_6)^!$ is generated by the vectors

$$(x_i \cdot x_j) \cdot x_k - x_i \cdot (x_j \cdot x_k),\ (x_i \cdot x_j) \cdot x_k - (x_j \cdot x_i) \cdot x_k,\ (x_i \cdot x_j) \cdot x_k - (x_i \cdot x_k) \cdot x_j.$$

But we can write

$$(x_i \cdot x_j) \cdot x_k - (x_j \cdot x_i) \cdot x_k = (Id - \tau_{ij})((x_i \cdot x_j) \cdot x_k)$$

and

$$(x_i \cdot x_j) \cdot x_k - (x_i \cdot x_k) \cdot x_j = (Id - \tau_{jk})((x_i \cdot x_j) \cdot x_k).$$

The $\Sigma_3$-invariant subspace of $\mathbb{K}[\Sigma_3]$ generated by the vectors $Id - \tau_{12}$ and $Id - \tau_{23}$ is of dimension 5, and from the classification [3], this space corresponds to $F_v = \mathbb{K}(\mathcal{O})(v)$ with

$$v = 2Id - \tau_{12} - \tau_{13} - \tau_{23} + c_1$$

and we deduce that this operad is of rank 2.

## 2. The 3-power associative algebras

We have seen that every $(G_i\text{-}\mathcal{A}ss)$-algebra is Lie-admissible. Moreover the operad $\mathcal{L}ie\mathcal{A}dm$ is quadratic, of rank 1. The submodule $R_6$ is of dimension 1 and corresponds to the one-dimensional $\Sigma_3$-invariant subspace $\mathbb{K}(\mathcal{O}(V)) = F_V$ of $\mathbb{K}[\Sigma_3]$, where $V$ is given by

$$V = \sum_{\sigma \in \Sigma_3} (-1)^{\epsilon(\sigma)} \sigma.$$

If we consider the natural action of $\Sigma_3$ on $\mathbb{K}[\Sigma_3]$, then there exists only two irreducible invariant one dimensional subspaces of $\mathbb{K}[\Sigma_3]$ that is $F_V$ and $F_W$ where

$$W = \sum_{\sigma \in \Sigma_3} \sigma.$$

If the set of Lie-admissible is associated to $F_V$, the set of algebras corresponding to $F_W$ is the set of 3-power associative algebras (see [3]) that is which satisfies $x^2 \cdot x = x \cdot x^2$ for every $x$. As in the Lie-admissible case we can define classes of 3-power associative algebras

corresponding to the action of the subgroups of $\Sigma_3$. We are conduced to consider the following submodules of $\Gamma(E)(3)$:

$$\begin{array}{rcl} R_1^{p^3} & = & R_1 \\ R_2^{p^3} & = & Vect_{\mathbb{K}}\{(x_i \cdot x_j)\cdot x_k - x_i\cdot(x_j\cdot x_k) + (x_j\cdot x_i)\cdot x_k - x_j\cdot(x_i\cdot x_k)\} \\ R_3^{p^3} & = & Vect_{\mathbb{K}}\{(x_i \cdot x_j)\cdot x_k - x_i\cdot(x_j\cdot x_k) + (x_i\cdot x_k)\cdot x_j - x_i\cdot(x_k\cdot x_j)\} \\ R_4^{p^3} & = & Vect_{\mathbb{K}}\{(x_i \cdot x_j)\cdot x_k - x_i\cdot(x_j\cdot x_k) + (x_k\cdot x_j)\cdot x_i - x_k\cdot(x_j\cdot x_i)\} \\ R_5^{p^3} & = & R_5 \\ R_6^{p^3} & = & Vect_{\mathbb{K}}\{(x_1\cdot x_2)\cdot x_3 - x_1\cdot(x_2\cdot x_3) + (x_2\cdot x_3)\cdot x_1 - x_2\cdot(x_3\cdot x_1) + (x_3\cdot x_1)\cdot x_2 \\ & & -x_3\cdot(x_1\cdot x_2) + (x_2\cdot x_1)\cdot x_3 - x_2\cdot(x_1\cdot x_3) + (x_3\cdot x_2)\cdot x_1 - x_3\cdot(x_2\cdot x_1) \\ & & +(x_1\cdot x_3)\cdot x_2 - x_1\cdot(x_3\cdot x_2)\} \end{array}$$

This corresponds to $R_i^{p^3} = \mathbb{K}(\mathcal{O}(Y_i))$ with

$$Y_i = \Sigma_{\sigma\in G_i}\sigma(X)$$

with $X = x_1\cdot(x_2\cdot x_3) - (x_1\cdot x_2)\cdot x_3$.
We denote by $(G_i{-}p^3\mathcal{A}ss)$ the corresponding quadratic operads. The corresponding dual operads are discribed by the following ideals of relations:

$$\begin{array}{rl} (R_1^{p^3})^! = & R_1 \\ (R_2^{p^3})^! = & Vect_{\mathbb{K}}\{(x_i\cdot x_j)\cdot x_k - x_i\cdot(x_j\cdot x_k); (x_i\cdot x_j)\cdot x_k + (x_j\cdot x_i)\cdot x_k\} \\ (R_3^{p^3})^! = & Vect_{\mathbb{K}}\{(x_i\cdot x_j)\cdot x_k - x_i\cdot(x_j\cdot x_k); (x_i\cdot x_j)\cdot x_k) + (x_i\cdot x_k)\cdot x_j\} \\ (R_4^{p^3})^! = & Vect_{\mathbb{K}}\{(x_i\cdot x_j)\cdot x_k - x_i\cdot(x_j\cdot x_k); (x_i\cdot x_j)\cdot x_k + (x_k\cdot x_j)\cdot x_i\} \\ (R_5^{p^3})^! = & R_5^! \\ (R_6^{p^3})^! = & Vect_{\mathbb{K}}\{(x_i\cdot x_j)\cdot x_k - x_i\cdot(x_j\cdot x_k); (x_i\cdot x_j)\cdot x_k + (x_j\cdot x_i)\cdot x_k \\ & (x_i\cdot x_j)\cdot x_k + (x_i\cdot x_k)\cdot x_j\}. \end{array}$$

The proof is analogous to the Lie-admissible case. Let us note that these operads are also of rank 2 except for $i = 1$.

### 3. The $\mathbb{K}[\Sigma_3]$-associative algebras

This example of nonassociative algebras generalizes the previous, considering not only the one dimensional invariant subspace of $\mathbb{K}[\Sigma_3]$ but all the invariant subspaces. Recall that, for every $v \in \mathbb{K}[\Sigma_3]$, we have denoted by $\mathcal{O}(v)$ the corresponding orbit and by $F_v = \mathbb{K}(\mathcal{O}(v))$ the linear subspace of $\mathbb{K}[\Sigma_3]$ generated by $\mathcal{O}(v)$. Since $F_v$ is a $\Sigma_3$-invariant subspace of $\mathbb{K}[\Sigma_3]$, by Mashke's theorem, it is a direct sum of irreducible invariant subspaces. Moreover, given an invariant subspace $F$ of $\mathbb{K}[\Sigma_3]$, there exists $v \in \mathbb{K}[\Sigma_3]$ (not necessarily unique) such that $F = F_v = \mathbb{K}(\mathcal{O}(v))$.

**Definition 5** *1. A $\mathbb{K}$-algebra $(\mathcal{A},\mu)$ is called $\mathbb{K}[\Sigma_3]$-associative if there exists $v \in \mathbb{K}[\Sigma_3], v \neq 0$, such that*

$$A(\mu)\circ\Phi_v^{\mathcal{A}} = 0$$

*where $A(\mu)$ is the associator of $\mu$.*

*2. Let $A(\mu)\circ\Phi_v^{\mathcal{A}} = 0$ and $A(\mu)\circ\Phi_w^{\mathcal{A}} = 0$ be two identities satisfied by the algebra $(\mathcal{A},\mu)$. We say that these identities are equivalent if $F_v = \mathbb{K}(\mathcal{O}(v)) = F_w = \mathbb{K}(\mathcal{O}(w))$.*

Remark that if $F_v$ is not an irreducible invariant subspace, then there exists $w \in F_v$ such that $F_w \subset F_v$ and $F_w \neq F_v$. In this case the identity $A(\mu) \circ \Phi_v^{\mathcal{A}} = 0$ implies $A(\mu) \circ \Phi_w^{\mathcal{A}} = 0$ but these identities are not equivalent.

**Examples.**
1. If $v = Id - \tau_{23}$, the relation

$$A(\mu) \circ \Phi_v^{\mathcal{A}} = 0 \tag{3}$$

becomes

$$A(\mu)(x_1 \otimes x_2 \otimes x_3 - x_1 \otimes x_3 \otimes x_2) = 0.$$

The corresponding algebra is a pre-Lie algebra.
2. The Lie-admissible and third-power associative algebras are $\mathbb{K}[\Sigma_3]$-associative algebras. In fact an algebra $(\mathcal{A}, \mu)$ is Lie-admissible if $A(\mu) \circ \Phi_V^{\mathcal{A}} = 0$ and third-power associative if $A(\mu) \circ \Phi_W^{\mathcal{A}} = 0$ with

$$V = Id - \tau_{12} - \tau_{23} - \tau_{13} + c_1 + c_2$$

and

$$W = Id + \tau_{12} + \tau_{23} + \tau_{13} + c_1 + c_2.$$

3. For $i = 1, ..., 6$ we denote by $V_i$ and $W_i$ the vectors of $\mathbb{K}[\Sigma_3]$ given by

$$V_i = \sum_{\sigma \in G_i} (-1)^{\epsilon(\sigma)} \sigma, \quad W_i = \sum_{\sigma \in G_i} \sigma.$$

Then $(\mathcal{A}, \mu)$ is a $G_i$-$\mathcal{A}ss$-algebra if

$$A(\mu) \circ \Phi_{V_i}^{\mathcal{A}} = 0$$

and a $G_i$-$p^3\mathcal{A}ss$-algebra if

$$A(\mu) \circ \Phi_{W_i}^{\mathcal{A}} = 0.$$

They are particular cases of $\mathbb{K}[\Sigma_3]$-associative algebras. We shall return, in the last section, on the determination of the corresponding operads.

# 3 The operad $\tilde{\mathcal{P}}$ associated to a quadratic operad $\mathcal{P}$

In the previous sections we saw that for some quadratic operads, the dual operad gives a way to construct on the tensor product $\mathcal{A} \otimes \mathcal{B}$ of a $\mathcal{P}$-algebra $\mathcal{A}$ and a $\mathcal{P}^!$-algebra $\mathcal{B}$ a structure of $\mathcal{P}$-algebra for the usual tensor product $\mu_{\mathcal{A} \otimes \mathcal{B}} = \mu_{\mathcal{A}} \otimes \mu_{\mathcal{B}}$. But this is not true for every quadratic operad. In this section we define from a given quadratic operad $\mathcal{P}$ an associated quadratic operad, denoted by $\tilde{\mathcal{P}}$, whose fondamental property is to satisfy the above property on the tensor product.

Let $(\mathcal{A}, \mu)$ be a $\mathcal{P}$-algebra where $\mathcal{P}$ is a quadratic operad. Let $R$ be the submodule of $\Gamma(E)(3)$ defining the relations of $\mathcal{A}$. We denote by $A^L(\mu) = \mu \circ (\mu \otimes Id)$ and $A^R(\mu) = \mu \circ (Id \otimes \mu)$. If we suppose that $R$ is of rank $k$, then the multiplication $\mu$ satisfies $k$ relations of type

$$A^L(\mu) \circ \Phi_{v_i}^{\mathcal{A}} - A^R(\mu) \circ \Phi_{w_i}^{\mathcal{A}} = 0$$

where $v_i, w_i \in \mathbb{K}[\Sigma_3]$ for any $i \in I = \{1,..,k\}$ and the vectors $(v_i)_{i\in I}$ are linearly independent as well as the vectors $(w_i)_{i\in I}$.

**Examples.**

1. The associative algebras correspond to $k = 1$ and $v_1 = w_1 = Id$, pre-Lie algebras to $k = 1$ and $v_1 = w_1 = Id - \tau_{23}$ and more generally $G_i$-associative algebras correpond to $k = 1$ and $v_1 = w_1 = V_i$. The Lie-admissible algebras correpond to $k = 1$, $v_1 = w_1 = V$ and the 3-power associative algebras to $v_1 = w_1 = W$.

2. The Leibniz algebras correspond to $k = 1$ and $v_1 = Id - \tau_{23}$, $w_1 = Id$.

Let $\mathcal{P}$ be a quadratic operad generated by $E \subset \mathbb{K}[\Sigma_2]$. For every $v = \Sigma_{l=1}^6 Sa_l\sigma_l \in \mathbb{K}[\Sigma_3]$ we consider on $\Gamma(E)(3)$ the linear maps

$$\Psi_v^L((x_i \cdot x_j) \cdot x_k) = \Sigma a_l\sigma_l((x_i \cdot x_j) \cdot x_k),\ \Psi_v^L(x_i \cdot (x_j \cdot x_k)) = 0$$

and

$$\Psi_v^R((x_i \cdot x_j) \cdot x_k) = 0,\ \Psi_v^R(x_i \cdot (x_j \cdot x_k)) = \Sigma a_l\sigma_l(x_i \cdot (x_j \cdot x_k)).$$

Let $R$ be the module of relations of $\mathcal{P}$. If it is of rank $k$, it is written

$$R = Vect_{\mathbb{K}}\left\{(\Psi_{v_p}^L((x_i \cdot x_j) \cdot x_k) - \Psi_{w_p}^R(x_1 \cdot (x_2 \cdot x_3)),\ 1 \leq p \leq k\right\}$$

with $v_p = \sum_{i=1}^6 a_i^p\sigma_i$ and $w_p = \sum_{i=1}^6 b_i^p\sigma_i$ for $1 \leq p \leq k$.
Let $\tilde{E}$ be the sub-module of $\mathbb{K}[\Sigma_2]$ defined by

$$\tilde{E} = \begin{cases} E \ \text{ if } E = \mathbb{1} \oplus Sgn_2 \\ \mathcal{C}om(2) = \mathbb{1} \ \text{ if } \ E = \mathbb{1} \ \text{ or } \ Sgn_2 \end{cases}$$

If $dim\tilde{E} = 2$, we denote $\tilde{R}$ the $\mathbb{K}[\Sigma_3]$-module generated by the vectors

$$\begin{cases} a_i^p a_j^p \Phi_{\sigma_i-\sigma_j}^L((x_1 \cdot x_2) \cdot x_3), \\ \\ b_i^p b_j^p \Phi_{\sigma_i-\sigma_j}^R(x_1 \cdot (x_2 \cdot x_3)), \\ \\ a_i^p b_j^p(\Phi_{\sigma_i}^L((x_1 \cdot x_2) \cdot x_3) - \Phi_{\sigma_j}^R(x_1 \cdot (x_2 \cdot x_3)), \end{cases}$$

for $1 \leq p \leq k$.
If $\tilde{E} = \mathcal{C}omm(2)$, $\tilde{R}$ is generated also by these vectors, but modulo the relations of commutation.

**Definition 6** *The operad $\tilde{\mathcal{P}}$ associated to the quadratic operad $\mathcal{P}$ is the quadratic operad generated by $\tilde{E}$ and with $\mathbb{K}[\Sigma_3]$-submodule of relations $\tilde{R}$.*

We have the main result :

**Theorem 7** *Let $\mathcal{A}$ be a $\mathcal{P}$-algebra and $\mathcal{B}$ a $\tilde{\mathcal{P}}$-algebra. Then the algebra $\mathcal{A} \otimes \mathcal{B}$ with product $\mu_{\mathcal{A}\otimes\mathcal{B}}$ is a $\mathcal{P}$-algebra.*

*Proof.* Let $\mathcal{A}$ be a $\mathcal{P}$-algebra. Its multiplication $\mu_{\mathcal{A}}$ satisfies

$$(A^L(\mu_{\mathcal{A}}) \circ \Phi^{\mathcal{A}}_{v_p} - A^R(\mu_{\mathcal{A}}) \circ \Phi^{\mathcal{A}}_{w_p})(x_1 \otimes x_2 \otimes x_3) = 0$$

for $p = 1, ..., k$. If $\mathcal{B}$ is a $\tilde{\mathcal{P}}$-algebra, its multiplication $\mu_{\mathcal{B}}$ satisfies

$$\begin{cases} A^L(\mu_{\mathcal{B}}) \circ \Phi^{\mathcal{B}}_{\sigma_i - \sigma_j}(y_1 \otimes y_2 \otimes y_3) = 0, \text{ if } \exists p,\ a^i_p a^j_p \neq 0 \\ A^R(\mu_{\mathcal{B}}) \circ \Phi^{\mathcal{B}}_{\sigma_i - \sigma_j}(y_1 \otimes y_2 \otimes y_3) = 0, \text{ if } \exists p,\ b^i_p b^j_p \neq 0 \\ A^L(\mu_{\mathcal{A}}) \circ \Phi^{\mathcal{B}}_{\sigma_i} - A^R(\mu_{\mathcal{B}}) \circ \Phi^{\mathcal{B}}_{\sigma_j}(y_1 \otimes y_2 \otimes y_3) = 0, \text{ if } \exists p,\ a^i_p b^j_p \neq 0. \end{cases}$$

Now we consider the product $\mu_{\mathcal{A}\otimes\mathcal{B}}$. We have

$$\begin{aligned} & (A^L(\mu_{\mathcal{A}\otimes\mathcal{B}}) \circ \Phi^{\mathcal{A}\otimes\mathcal{B}}_{v_p} - A^R(\mu_{\mathcal{A}\otimes\mathcal{B}}) \circ \Phi^{\mathcal{A}\otimes\mathcal{B}}_{w_p})(x_1 \otimes y_1 \otimes x_2 \otimes y_2 \otimes x_3 \otimes y_3) \\ = & (\Sigma a^p_i A^L(\mu_{\mathcal{A}\otimes\mathcal{B}}) \circ \sigma_i - \Sigma b^p_i A^R(\mu_{\mathcal{A}\otimes\mathcal{B}}) \circ \sigma_i)(x_1 \otimes y_1 \otimes x_2 \otimes y_2 \otimes x_3 \otimes y_3) \\ = & \Sigma a^p_i (A^L(\mu_{\mathcal{A}}) \circ \sigma_i(x_1 \otimes x_2 \otimes x_3) \otimes A^L(\mu_{\mathcal{B}}) \circ \sigma_i(y_1 \otimes y_2 \otimes y_3)) \\ & - \Sigma b^p_i (A^R(\mu_{\mathcal{A}}) \circ \sigma_i(x_1 \otimes x_2 \otimes x_3) \otimes A^R(\mu_{\mathcal{B}}) \circ \sigma_i(y_1 \otimes y_2 \otimes y_3)) \\ = & (\Sigma a^p_i (A^L(\mu_{\mathcal{A}}) \circ \sigma_i(x_1 \otimes x_2 \otimes x_3)) \otimes A^L(\mu_{\mathcal{B}}) \circ \sigma_j(y_1 \otimes y_2 \otimes y_3) \\ & -(\Sigma b^p_i (A^R(\mu_{\mathcal{A}}) \circ \sigma_i(x_1 \otimes x_2 \otimes x_3)) \otimes A^R(\mu_{\mathcal{B}}) \circ \sigma_j(y_1 \otimes y_2 \otimes y_3)) \end{aligned}$$

where $j$ is choosen in $\{1, \cdots 6\}$ such that $a^p_j \neq 0$,

$$\begin{aligned} = & \ \Sigma a^p_i A^L(\mu_{\mathcal{A}}) \circ \sigma_i(x_1 \otimes x_2 \otimes x_3) - \Sigma b^p_i A^R(\mu_{\mathcal{A}}) \circ \sigma_i(x_1 \otimes x_2 \otimes x_3)) \\ & \otimes A^R(\mu_{\mathcal{B}}) \circ \sigma_j(y_1 \otimes y_2 \otimes y_3)) \\ = & \ (A^L(\mu_{\mathcal{A}}) \circ \Phi^{\mathcal{A}}_{v_p} - A^R(\mu_{\mathcal{A}}) \circ \Phi^{\mathcal{A}}_{w_p})(x_1 \otimes x_2 \otimes x_3)) \otimes A^R(\mu_{\mathcal{B}}) \circ \sigma_j(y_1 \otimes y_2 \otimes y_3)) \\ = & \ 0. \end{aligned}$$

# 4 Some examples

## 4.1 $\mathcal{P} = G_i{-}\mathcal{A}ss$

**Proposition 8** *If $\mathcal{P}$ is a $(G_i{-}\mathcal{A}ss)$ operad, then the operads $\mathcal{P}^!$ and $\tilde{\mathcal{P}}$ are equal.*

*Proof.* In this case we have $k = 1$ and $v_1 = w_1 = V_i$. Then $\tilde{\mathcal{P}}$ is defined by the module of relations

$$\begin{cases} a_i a_j \Psi^L_{\sigma_i - \sigma_j}((x_1 \cdot x_2) \cdot x_3), \\ a_i a_j \Psi^R_{\sigma_i - \sigma_j}(x_1 \cdot (x_2 \cdot x_3)), \\ a_i a_j (\Psi^L_{\sigma_i}((x_1 \cdot x_2) \cdot x_3) - \Psi^R_{\sigma_j}(x_1 \cdot (x_2 \cdot x_3)), \end{cases}$$

where $V_i = \Sigma a_j \sigma_j$. This system is reduced to

$$\{\ \Psi^L_{Id}((x_1 \cdot x_2) \cdot x_3) - \Psi^R_{Id}(x_1 \cdot (x_2 \cdot x_3)), a_i a_j \Psi^L_{\sigma_i - \sigma_j}((x_1 \cdot x_2) \cdot x_3),$$

which corresponds to the dual operad.
In particular, if $\mathcal{P} = \mathcal{L}ie\mathcal{A}dm$, then $\mathcal{P} = \tilde{\mathcal{P}} = \mathcal{C}omm3$ where $\mathcal{C}omm3$ is the quadratic operad defined from the submodule of relations :

$$R = Vect_{\mathbb{K}}\{(x_i \cdot x_j) \cdot x_k - x_i \cdot (x_j \cdot x_k), (x_i \cdot x_j) \cdot x_k - (x_j \cdot x_i) \cdot x_k\}.$$

Thus, a $\mathcal{C}omm3$-algebra $\mathcal{A}$ is 3-commutative if it is associative and satisfies

$$x_i \cdot x_j \cdot x_k = x_{\sigma(i)} \cdot x_{\sigma(j)} \cdot x_{\sigma(k)}$$

for every $\sigma \in \Sigma_3$. If $\mathcal{A}$ is unitary this implies that $\mathcal{A}$ is a commutative algebra. If not, we have that $\mathcal{A}^2$ is contained in the center of $\mathcal{A}$. The associated Lie algebra is two step nilpotent.

## 4.2 $\mathcal{P} = \mathcal{L}ie$

If $\mathcal{P} = \mathcal{L}ie$ then $\tilde{\mathcal{P}} = \mathcal{P}^! = \mathcal{C}om$. In fact, in this case, $k = 1$ and $v_1 = w_1 = Id + c_1 + c_2$. As $v_1 = w_1$, a $\tilde{\mathcal{P}}$-algebra is associative and the module of relation corresponds to the operad $\mathcal{C}om$.

## 4.3 $\mathcal{P} = \mathcal{L}eib$

Let $\mathcal{P} = \mathcal{L}eib$ be the Leibniz operad. A Leibniz algebra is defined by the relation

$$x(yz) - (xy)z + (xz)y = 0.$$

In this case the associated $\tilde{\mathcal{P}}$ operad corresponds to the relations

$$x(yz) = (xy)z$$

and

$$(xy)z = (xz)y.$$

Thus a $\tilde{\mathcal{L}eib}$-algebra is an associative algebra satisfying

$$xyz = xzy.$$

This relation is equivalent to

$$x[y, z] = 0$$

with $[y, z] = yz - zy$. This last identity implies that if $x$ et $y$ are in the derived Lie subalgebra of the associated Lie algebra, then $xy = 0$. The derived Lie algebra is then abelian and the Lie algebra is 2 step nilpotent. The dual operad, also denoted by $\mathcal{Z}inb$, corresponds to the identity

$$(xy)z - x(yz) - x(zy) = 0.$$

Thus a $\tilde{\mathcal{L}eib}$-algebra is a Zinbiel algebra (i.e. a $\mathcal{L}eib^!$-algebra) if $x(yz) = (xy)z = 0$ (every product of 3 elements of the associative algebra is zero). These algebras are nilalgebras $\mathcal{A}$ satisfying $\mathcal{A}^3 = 0$. For example, any associative commutative algebra is a $\tilde{\mathcal{L}eib}$-algebra. Every $\tilde{\mathcal{L}eib}$-algebra with unit is commutative. In dimension 3 the algebra defined by

$$e_1e_1 = e_2, \; e_1e_3 = e_3e_3 = e_2$$

is a noncommutative $\tilde{\mathcal{L}eib}$-algebra.

## 4.4 Determination of $\tilde{\mathcal{P}}$ in the Lie-admissible case

In the following table we describe the multiplications of the algebras corresponding to the operads $\mathcal{P}, \mathcal{P}^!, \tilde{\mathcal{P}}$ and this for the operads described in [3], Theorem 3. Recall that these algebras are Lie-admissible $\mathbb{K}[\Sigma_3]$-associatives algebras. We denote by $A(x,y,z)$ the associator : $A(x,y,z) = (x\cdot y)\cdot z - x\cdot(y\cdot z)$.

$$\begin{cases} \mathcal{P} = \mathcal{L}ie\mathcal{A}dm : A(x,y,z) - A(y,x,z) - A(x,z,y) - A(z,y,x) + A(y,z,x) + A(z,x,y) = 0. \\ \mathcal{P}^! : A(x,y,z) = 0,\ x\cdot y\cdot z = y\cdot x\cdot z = x\cdot z\cdot y. \\ \tilde{\mathcal{P}} = \mathcal{P}^! \end{cases}$$

$$\begin{cases} \mathcal{P} = G_5 - \mathcal{A}ss : A(x,y,z) + A(y,z,x) + A(z,x,y) = 0. \\ \mathcal{P}^! : A(x,y,z) = 0,\ x\cdot y\cdot z = y\cdot z\cdot x = z\cdot x\cdot y. \\ \tilde{\mathcal{P}} = \mathcal{P}^! \end{cases}$$

$$\begin{cases} \mathcal{P} : \alpha A(x,y,z) - \alpha A(y,x,z) + (\alpha+\beta-3)A(z,y,x) - \beta A(x,z,y) + \beta A(y,z,x) \\ \quad +(3-\alpha-\beta)A(z,x,y) = 0,\ (\alpha,\beta) \neq (1,1) \\ \mathcal{P}^! : A(x,y,z) = 0, (\alpha-\beta)(x\cdot y\cdot z - y\cdot x\cdot z) + (\alpha+2\beta-3)(z\cdot y\cdot x - z\cdot x\cdot y) = 0 \end{cases}$$

The computation of $\tilde{\mathcal{P}}$ depends on the values of the parameters $\alpha$ et $\beta$. If $(\alpha,\beta) \neq (3,0)$ ou $(0,3)$ or $(0,0)$ then

$$\tilde{\mathcal{P}} = \mathcal{L}ie\mathcal{A}dm^!$$

If $(\alpha,\beta) = (3,0)$ then $\mathcal{P} = G_2 - \mathcal{A}ss$ and

$$\tilde{\mathcal{P}} = \mathcal{P}^! = G_2 - \mathcal{A}ss^!.$$

If $(\alpha,\beta) = (0,3)$ then $\mathcal{P} = G_4 - \mathcal{A}ss$ and

$$\tilde{\mathcal{P}} = \mathcal{P}^! = G_4 - \mathcal{A}ss^!.$$

If $(\alpha,\beta) = (0,0)$, then $\mathcal{P} = G_3 - \mathcal{A}ss$ and

$$\tilde{\mathcal{P}} = \mathcal{P}^! = G_3 - \mathcal{A}ss^!.$$

$$\begin{cases} \mathcal{P} : A(x,y,z) + (1+t)A(y,x,z) + A(z,y,x) + A(y,z,x) + (1-t)A(z,x,y) = 0, t \neq 1 \\ \mathcal{P}^! : A(x,y,z) = 0, (t-1)x\cdot y\cdot z - (t-1)y\cdot x\cdot z - (t+2)z\cdot y\cdot x + (1+2t)x\cdot z\cdot y \\ \quad -(1+2t)y\cdot z\cdot x + (t+2)z\cdot x\cdot y = 0. \\ \tilde{\mathcal{P}} = \mathcal{L}ie\mathcal{A}dm^!. \end{cases}$$

$$\begin{cases} \mathcal{P} : 2A(x,y,z) + A(y,x,z) + A(x,z,y) + A(y,z,x) + A(z,x,y) = 0. \\ \mathcal{P}^! : A(x,y,z) = 0, x\cdot y\cdot z + y\cdot x\cdot z - z\cdot y\cdot x - z\cdot x\cdot y = 0. \\ \tilde{\mathcal{P}} = \mathcal{L}ie\mathcal{A}dm^!. \end{cases}$$

$$\begin{cases} \mathcal{P} : 2A(x,y,z) - A(y,x,z) - A(z,y,x) - A(x,z,y) + A(y,z,x) = 0. \\ \mathcal{P}^! : A(x,y,z) = 0, x\cdot y\cdot z - y\cdot x\cdot z - z\cdot y\cdot x - x\cdot z\cdot y + y\cdot z\cdot x + z\cdot x\cdot y = 0 \\ \tilde{\mathcal{P}} = \mathcal{L}ie\mathcal{A}dm^!. \end{cases}$$

$$\begin{cases} \mathcal{P} = \mathcal{A}ss : A(x,y,z) = 0. \\ \mathcal{P}^! = \mathcal{A}ss \\ \tilde{\mathcal{P}} = \mathcal{A}ss \end{cases}$$

**Proposition 9** *Let $\mathcal{P}$ be an operad corresponding to a $\mathbb{K}[\Sigma_3]$-associative Lie-admissible algebra type. Then $\tilde{\mathcal{P}} = \mathcal{P}^!$ if and only if $\mathcal{P}$ is the operad $G_i - \mathcal{A}ss$ for some $i$.*

We can also find the operads such that $\tilde{\mathcal{P}} = \mathcal{P}^!$ in the case of a quadratic operad generated by a commutative operation (i.e $E = 1\!\!1$) or an anticommutative one (i.e $E = Sgn_2$)

**Proposition 10** *Let $\mathcal{P} = \mathcal{P}(\mathbb{K}, E, R)$ be a quadratic operad generated by an operation with a symmetry (i.e. $E = Sgn_2$ or $E = 1\!\!1$). Then $\tilde{\mathcal{P}} = \mathcal{P}^!$ if and only if $E = Sgn_2$ and $\mathcal{P} = \mathcal{C}om$ or $\mathcal{P} = \Gamma(Sgn_2)$, the free operad generated by the signum representation.*

## 4.5 $\mathcal{P} = \mathcal{P}oiss$

A Poisson algebra over $\mathbb{K}$ is a $\mathbb{K}$-vector space equipped with two bilinear products:
1) a Lie algebra multiplication, denoted by $\{,\}$, called the Poisson bracket,
2) an associative commutative multiplication, denoted by $\bullet$.
These two operations satisfy the Leibniz condition:

$$\{X \bullet Y, Z\} = X \bullet \{Y, Z\} + \{X, Z\} \bullet Y, \tag{4}$$

for all $X, Y, Z$. In [7], one proves that a Poisson algebra can be defined by only one nonassociative product, denoted by $X \cdot Y$, satisfying the following identity

$$3A.(X, Y, Z) = (X \cdot Z) \cdot Y + (Y \cdot Z) \cdot X - (Y \cdot X) \cdot Z - (Z \cdot X) \cdot Y, \tag{5}$$

where $A.(X, Y, Z) = (X \cdot Y) \cdot Z - X \cdot (Y \cdot Z)$ is the associator of the product $\cdot$. The corresponding quadratic operad is with one generating operation and of rank 1. Let us denote by $\mathcal{P}oiss$ this operad. If

$$\Phi^L_{v_1}((x_i \cdot x_j) \cdot x_k) - \Phi^R_{w_1}(x_1 \cdot (x_2 \cdot x_3))$$

is the generator of the module of relations $R$ of $\mathcal{P}oiss$, we have

$$v_1 = 3Id - \tau_{23} - c_1 + \tau_{12} + c_2$$

and

$$w_1 = 3Id.$$

Then $\tilde{\mathcal{P}oiss}$ is generated by

$$\left\{ \begin{array}{l} (x_1 \cdot x_2) \cdot x_3 - (x_1 \cdot x_3) \cdot x_2 \\ (x_1 \cdot x_2) \cdot x_3 - (x_2 \cdot x_3) \cdot x_1 \\ (x_1 \cdot x_2) \cdot x_3 - (x_2 \cdot x_1) \cdot x_3 \\ (x_1 \cdot x_2) \cdot x_3 - (x_3 \cdot x_1) \cdot x_2 \\ (x_1 \cdot x_2) \cdot x_3 - x_1 \cdot (x_2 \cdot x_3) \end{array} \right.$$

and $\tilde{\mathcal{P}oiss} = \mathcal{C}omm3$.

**Remark.** In this work, we have considered only the classical product on the tensor product of algebras. It is possible to do the same work considering generalized or twisted tensor product. In these case we can probably define also an associated operad.

Let us consider two Poisson algebras $(\mathcal{A}, \mu_{\mathcal{A}})$ and $(\mathcal{B}, \mu_{\mathcal{B}})$ two Poisson algebras defined by the nonassociative multiplication (5). Let $\tau$ be the twist map:

$$\tau(x \otimes y) = y \otimes x.$$

If we consider on $\mathcal{A} \otimes \mathcal{B}$ the following product

$$\mu_{\mathcal{A}} \otimes_{\tau} \mu_{\mathcal{B}} = 3\mu_{\mathcal{A}} \otimes \mu_{\mathcal{B}} - \mu_{\mathcal{A}} \otimes (\mu_{\mathcal{B}} \circ \tau) - (\mu_{\mathcal{A}} \circ \tau) \otimes \mu_{\mathcal{B}} + (\mu_{\mathcal{A}} \circ \tau) \otimes (\mu_{\mathcal{B}} \circ \tau)$$

then $(\mathcal{A} \otimes \mathcal{B}, \mu_{\mathcal{A}} \otimes_{\tau} \mu_{\mathcal{B}})$ is a Poisson algebra.

# References


[1] Ginzburg V., Kapranov M., Koszul duality for operads. *Duke Math Journal.* **76** 1, (1994), 203-272.

[2] Goze M., Remm E., Lie-admissible algebras and operads. *Journal of Algebra* **273/1** (2004), 129-152.

[3] Goze M., Remm E., A class of nonassociative algebras. Algebra colloquium 2007. To appear

[4] Loday J.-L., Une version non commutative des algèbres de Lie: les algèbres de Leibniz. *Enseign. Math.* (2) 39 (1993) n.3-4, 269-293.

[5] Markl, M., Homotopy algebras are homotopy algebras. Forum Math. 16 (2004), no. 1, 129–160.

[6] Markl M., Shnider S., Stasheff J., *Operads in algebra, topology and physics.* Mathematical Surveys and Monographs, 96. American Mathematical Society, Providence, RI, 2002. x+349 pp.

[7] Markl M., Remm E., Algebras with one operation including Poisson and other Lie-admissible algebras. To appear in *Journal of Algebra.*